Local monomialization conjecture of a singular foliation of Darboux type

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Abstract

After the nice result introduced by Belotto in [1] concerning the local monomialization of a singular foliation given by n first integrals, this work is a continuation in the same spirit. In this paper, we introduce a important conjecture about local monomialization of a singular foliation of Darboux type (see section 1). This conjecture can be used to study pseudo-abelian integrals [2,4].

1 Introduction

Let M be an analytic manifold of dimension $n+2$. Given a families of first integrals of Darboux type H_ϵ

$$H_\epsilon(x,y) = H(x,y,\epsilon_1,\dots,\epsilon_n) = \prod_{i=1}^{k} P^{a_i}(x,y,\epsilon_1,\dots,\epsilon_n), \quad a_i > 0. \tag{1}$$

Let F be the foliation of codimension one in M with coordinates $(x,y,\epsilon_1,\dots,\epsilon_n)$ which is given by the analytic one form ω

$$\omega = \frac{H_x}{\phi}dx + \frac{H_y}{\phi}dy + \sum_{i=1}^{n}\frac{H_{\epsilon_i}}{\phi}d\epsilon_i = 0, \tag{2}$$

where $H_x = \frac{\partial H}{\partial x}, H_y = \frac{\partial H}{\partial y}, H_{\epsilon_i} = \frac{\partial H}{\partial \epsilon_i}$ and $\phi = \prod_{i=1}^{k} P_i^{a_i-1}(x,y,\epsilon_1,\dots,\epsilon_n)$ (integrating factor).

Let $F_i, i = 1,\dots,n$ are foliations of codimension one in M with coordinates $(x,y,\epsilon_1,\dots,\epsilon_n)$ which are given by the one forms ω_i

$$\omega_i = d\epsilon_i = 0, \quad i = 1,\dots,n. \tag{3}$$

Let $\mathcal{F} = (F, F_1,\dots,F_n)$ be the result foliation of dimension one in M where its leaves are given by the transversal intersection of leaves of $F, F_1,\dots,F_n$. Otherwise speaking, the singular foliation $\mathcal{F}$ is given by

$$\begin{aligned}\Omega &= \omega \wedge \omega_1 \wedge \dots \wedge \omega_n & (4)\\ &= Q_1(x,y,\epsilon_1,\dots,\epsilon_n)dx \wedge d\epsilon_1 \wedge \dots \wedge d\epsilon_n + Q_2(x,y,\epsilon_1,\dots,\epsilon_n)dy \wedge d\epsilon_1 \wedge \dots \wedge d\epsilon_n = 0, & (5)\end{aligned}$$

where $Q_1 = \frac{H_x}{\phi}, Q_2 = \frac{H_y}{\phi}$ are polynomials.

We shall say that Ω is a foliation of Darboux type with first integrals $(H,\epsilon_1,\dots,\epsilon_n)$.

Example. Let $H_\epsilon(x,y) = H(x,y,\epsilon) = (x-\epsilon)^{a_1}(x-y)^{a_2}(x+y)^{a_3}$ be a the first integral of Darboux type. The foliation F of codimension one in three dimensional space M with coordinates (x,y,ϵ) is given by the one form

$$\begin{aligned}\omega = &(a_1(x-y)(x+y) + a_2(x-\epsilon)(x+y) + a_3(x-\epsilon)(x-y))dx\\ &-(a_2(x-\epsilon)(x+y) - a_3(x-\epsilon)(x-y))dy - a_1(x-y)(x+y)d\epsilon = 0\end{aligned}$$

and the foliation F_1 of codimension one in M is given by the one form

$$\omega_1 = d\epsilon = 0.$$

The result foliation $\mathcal{F}=(F,F_1)$ is given by the two-form

$$\Omega=\omega\wedge\omega_1=(a_1(x-y)(x+y)+a_2(x-\epsilon)(x+y)+a_3(x-\epsilon)(x-y))dx\wedge d\epsilon-(a_2(x-\epsilon)(x+y)-a_3(x-\epsilon)(x-y))dy\wedge d\epsilon=0$$

Observe that the foliation $\mathcal{F}=(F,F_1)$ has a complicated singularity at the origin $(0,0,0)\subset D_0=\{\epsilon=0\}$.

Conjecture. There exist sequences of local blowings-up such that the total transform of the foliation $\mathcal{F}:\omega\wedge\omega_1\wedge\ldots\wedge\omega_n=0$ has locally $n+1$ monomial first integrals $(z^{\gamma_0},z^{\gamma_1}...,z^{\gamma_n})$ where $z^{\gamma_i}=z_1^{\gamma_{i,1}}\cdots z_{n+2}^{\gamma_{i,n+2}}$ and the exponents matrix

$$m(a_1,\ldots,a_k)=\begin{pmatrix}\gamma_0^1 & \cdots & \gamma_0^{n+2}\\ \gamma_1^1 & \cdots & \gamma_1^{n+2}\\ \vdots & \vdots & \vdots\\ \gamma_n^1 & \cdots & \gamma_n^{n+2}\end{pmatrix}$$

has a maximal rank.

2 Blowing-up of the foliation $\mathcal{F}$

In this section, we introduce the fundamental idea to prove the conjecture which is based in first step on Hironaka's reduction of singularities [3]. Let $D_0=\{\epsilon_1=\epsilon_2=\ldots=\epsilon_n=0\}$ be a initial exceptional divisor.

Theorem 1. *There exist a morphism Φ such that the pull-back foliation $\widetilde{\Phi}^*\mathcal{F}=\widetilde{\mathcal{F}}$ is given locally in neighborhood U_1 of the divisor $\widetilde{\Phi}^*(D_0)$ with coordinates $z=(z_1,\ldots,z_{n+2})$ by the following system*

$$\begin{cases}\widetilde{H}=z^{\gamma_0}.\Delta_0,\\ \tilde{\epsilon}_1=z^{\gamma_1}.\Delta_1,\\ \vdots\\ \tilde{\epsilon}_n=z^{\gamma_n}.\Delta_n,\end{cases}\tag{6}$$

where $\Delta_i, i=0,\ldots,n$ are a units.

Proof. (1) In first step, we monomialize the principal ideal $I_1=<P_1>$, Hironaka theorem's guarantee the existence oft a sequence of blow-ups $\widetilde{\Phi}_1=\widetilde{\Phi}^1_{n_1}\circ\widetilde{\Phi}^1_{n_1-1}\circ\ldots\circ\widetilde{\Phi}^1_1$ with initial center $C_0\subset D_0$ (which is possibly a submanifold of M) such that

$$(\widetilde{\Phi}_1^*P_1)^{a_1}=\delta_1\prod_{i=1}^{n+2}z_i^{a_1\tilde{\beta}_i^1},\quad \delta_1(0)\neq 0.$$

(2) In the second step, we consider the principal ideal $I_2=<\widetilde{\Phi}_1^*P_2>$ and by Hironaka theorem's there exist a sequence of blow-ups $\widetilde{\Phi}_2=\Phi^2_{n_2}\circ\Phi^2_{n_2-1}\circ\ldots\circ\Phi^2_1$ such that

$$(\widetilde{\Phi}_2^*\circ\widetilde{\Phi}_1^*P_2)^{a_2}=\delta_2\prod_{i=1}^{n+2}z_i^{a_2\tilde{\beta}_i^2},\quad \delta_2(0)\neq 0.$$

In the k-th step there exist a sequence of blow-ups $\widetilde{\Phi}_k=\Phi^k_{n_k}\circ\Phi^k_{n_k-1}\circ\ldots\circ\Phi^k_1$ such that the principal ideal $I_k=<\widetilde{\Phi}_{k-1}^*\circ\widetilde{\Phi}_{k-2}^*\circ\ldots\circ\widetilde{\Phi}_1^*P_k>$ has a normal crossings i.e.

$$(\widetilde{\Phi}_{k-1}^*\circ\widetilde{\Phi}_{k-2}^*\circ\ldots\circ\widetilde{\Phi}_1^*P_k)^{a_k}=\delta_k\prod_{i=1}^{n+2}z_i^{a_k\tilde{\beta}_i^k},\quad \delta_k(0)\neq 0.$$

Finally, after desingularisation of each polynomial P_i of the first integral $H=\prod_{i=1}^k P_i^{a_i}$, the equations $z_1=0,\ldots,z_{n+2}=0$ are corresponding the irreducibles components of the exceptional divisor. For this

raison after desingularisation of $\widetilde{\Phi}^*_{i-1}\circ\widetilde{\Phi}^*_{i-2}\circ\ldots\circ\widetilde{\Phi}^*_1 P_i$, the polynomial $\widetilde{\Phi}^*_i\circ\widetilde{\Phi}^*_{i-1}\circ\widetilde{\Phi}^*_{i-2}\circ\ldots\circ\widetilde{\Phi}^*_1 P_{i-1}$ has a normal crossings. So locally we have

$$\begin{cases} \widetilde{H} = z^{\gamma_0}.\Delta_0, \\ \tilde{\epsilon}_1 = z^{\gamma_1}.\Delta_1, \\ \vdots \\ \tilde{\epsilon}_n = z^{\gamma_n}.\Delta_n, \end{cases}$$

where $z=(z_1,\ldots,z_{n+2}), \gamma_0=\sum_{i=1}^k a_i\beta_i, \beta_i=(\beta_i^1,\ldots,\beta_i^{n+2}), \gamma_i=(\gamma_i^1,\ldots,\gamma_i^{n+2})$. $\square$

To complete the proof its necessairy to eliminate the units $\Delta_0,\Delta_1,\ldots,\Delta_n$ in the system (6). Now we define the resonant locus of the foliation $(z^{\gamma_0}.\Delta_0, z^{\gamma_1}.\Delta_1,\ldots,z^{\gamma_n}.\Delta_n)$

$$\mathcal{R} := \{a=(a_1,\ldots,a_k) : \gamma_0\wedge\sum_{j=1}^n\gamma_j = 0\}.$$

To prove the conjecture, we distinguish two cases

- **generic case** $a\notin\mathcal{R}$.
- **nongeneric case** $a\in\mathcal{R}$.

3 Some examples in dimension three

To more understand the problem, we see some examples in dimension three.

Example 1: Let $\mathcal{F}$ be the local foliation which is obtained by after k blow-ups. The foliation $\mathcal{F}$ is given by the following system

$$\begin{cases} H_a = x^{a_1}y^{a_2}(1+z) \\ f = xy \end{cases} \tag{7}$$

In this example we have $\gamma_0 : a_1\beta_1 + a_2\beta_2$ where $\beta_1=(1,0,0), \beta_2=(0,1,0)$, $\gamma_1=(1,1,0)$ and $\mathcal{R}=\{a=(a_1,a_2):\gamma_0\wedge\gamma_1=0\}$. Our goal is to kill the unit $1+z$ in the first integral H_a without modifying the second monomial f in the sense to preserve its monomiality structure. For this raison, we distinguish two different cases:

(a) The generic case $a_1\neq -a_2 \Leftrightarrow a\notin\mathcal{R}$: We take the change of variables $\tilde{x}=x(1+z)^{\frac{1}{a_1+a_2}}, \tilde{y}=y(1+z)^{\frac{1}{a_2+a_1}}$ and $\tilde{z}=z$. Then, we obtain the following system

$$\begin{cases} H_a = \tilde{x}^{a_1}\tilde{y}^{a_2} \\ f = \tilde{x}\tilde{y} \end{cases} \tag{8}$$

Question: How to calculate the generator vector field of the monomial foliation (7)?

Let us assume that the foliation $\mathcal{F}$ is generated locally by the vector field $X(\tilde{x},\tilde{y},z)=\alpha_1\tilde{x}\frac{\partial}{\partial\tilde{x}}+\alpha_2\tilde{y}\frac{\partial}{\partial\tilde{y}}+\alpha_3 z\frac{\partial}{\partial z}$ which satisfies

$$X(H)=X(\tilde{x}^{a_1}\tilde{y}^{a_2})=0, \qquad X(f)=X(\tilde{x}\tilde{y})=0$$

To determine the vector $\alpha=(\alpha_1,\alpha_2,\alpha_3)$ we use the two following relations

$$<\alpha,\gamma_0>=0 \quad \text{(i.e.} \quad X(H)=X(\tilde{x}^{a_1}\tilde{y}^{a_2})=0), \quad <\alpha,\gamma_1>=0 \quad \text{(i.e.} \quad X(f)=X(\tilde{x}\tilde{y})=0).$$

where $<,>$ is scalar product in $\mathbb{C}^3$. Finally, the vector $(\alpha_1,\alpha_2,\alpha_3)\in\{e_3\}$ and then

$$\mathcal{F}=\{z\frac{\partial}{\partial z}\}$$

Now, we express the vector field X in the original coordinates (x,y,z). If we write $X(x,y,z) = Ax\frac{\partial}{\partial x} + By\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$, to determine A, B we use the fact that

$$X(xy) = 0 \Longleftrightarrow A = -B$$

and so $X(x,y,z) = A(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}) + z\frac{\partial}{\partial z}$ on the other hand we have

$$Ax = X(x) = X(\tilde{x}(1+z)^{\frac{1}{a_1+a_2}}) = z\frac{\partial}{\partial z}(\tilde{x}(1+z)^{\frac{1}{a_1+a_2}}) = \tilde{x}(1+z)^{\frac{1}{a_1+a_2}-1}\frac{z}{a_1+a_2}$$

Finally, we obtain

$$X(x,y,z) = \frac{1}{a_1+a_2}\frac{z}{1+z}(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}) + z\frac{\partial}{\partial z} \Rightarrow Y(x,y,z) = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} + (a_1+a_2)(1+z)\frac{\partial}{\partial z}$$

Remark 1. *In dimension three, if we consider the foliation $\mathcal{F}$ which is given locally by*

$$\begin{cases} f_1 = x^a y^b z^c \\ f_2 = x^{\tilde{a}} y^{\tilde{b}} z^{\tilde{c}} \end{cases}$$

where $\mathrm{rank}\begin{pmatrix} a & b & c \\ \tilde{a} & \tilde{b} & \tilde{c} \end{pmatrix} = 2$. *The generator vector field X of the form*

$$X(x,y,z) = \hat{a}x\frac{\partial}{\partial x} + \hat{b}y\frac{\partial}{\partial y} + \hat{c}z\frac{\partial}{\partial z},$$

where

$$< (\hat{a},\hat{b},\hat{c}),(a,b,c) >= 0, \quad \textit{and} \quad < (\hat{a},\hat{b},\hat{c}),(\tilde{a},\tilde{b},\tilde{c}) >= 0.$$

In our example we observe that in the neighborhood of the leaf $\{z=0\}$ the vector field

$$Y \simeq x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} + (a_1+a_2)z\frac{\partial}{\partial z}$$

is linearizable and consequently Y is transversal to the leaf $\{z=0\}$.
(b) The problem suppose where $a_1 = -a_2$ i.e $a \in \mathcal{R} = \{a = (a_1,a_2) : \gamma_0 \wedge \gamma_1 = 0\}$. In this case near the leaf $\{z=0\}$, we have

$$Y \simeq x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}.$$

It is clear that the condition of transversality of Y and the leaf $\{z=0\}$ is not satisfieted.

Example 2: let $\mathcal{F}$ be the local foliation which is given (after a sequence of blow-ups) by

$$\begin{cases} H = x^{a_1}y^{a_2}z^{a_3}(1+g(x,y,z)) \\ f = xyz. \end{cases}$$

The foliation $\mathcal{F}$ is given also

$$\begin{cases} \frac{H}{f^{a_1}} = y^{a_2-a_1}z^{a_3-a_1}(1+g(x,y,z)) \\ f = xyz. \end{cases}$$

If $a = (a_1,a_2,a_3) \notin \mathcal{R} = \{a : (a_1\beta_1 + a_2\beta_2 + a_3\beta_3) \wedge (1,1,1) = 0\}$ (resonant locus), we can take the following variables change $x = \tilde{x}, y = \tilde{y}(1+g(x,y,z))^{\frac{1}{a_2-a_3}}$ and $z = \tilde{z}(1+g(x,y,z))^{\frac{1}{a_3-a_2}}$ and in this case the local foliation $\mathcal{F}$ is generated by the vector field

$$X(\tilde{x},\tilde{y},\tilde{z}) = (a_2-a_3)\tilde{x}\frac{\partial}{\partial \tilde{x}} + (a_3-a_1)\tilde{y}\frac{\partial}{\partial \tilde{y}} + (a_1-a_2)\tilde{z}\frac{\partial}{\partial \tilde{z}}$$

let us express the vector field X in the original coordinates (x, y, z), so we have

$$X(x) = X(\tilde{x}) = (a_2 - a_3)x\frac{\partial}{\partial x}$$

$$X(\tilde{y}(1 + g(x, y, z)^{\frac{1}{a_2 - a_3}})) = (a_3 - a_1)y + \frac{1}{a_2 - a_3}y\frac{X(g(x, y, z))}{1 + g(x, y, z)}$$

and

$$X(\tilde{z}(1 + g(x, y, z)^{\frac{1}{a_3 - a_2}})) = (a_1 - a_2)z + \frac{1}{a_3 - a_2}z\frac{X(g(x, y, z))}{1 + g(x, y, z)}.$$

Finally the vector field X of the form

$$X(x, y, z) = (a_2 - a_3)x\frac{\partial}{\partial x} + (a_3 - a_1)y\frac{\partial}{\partial y} + (a_1 - a_2)z\frac{\partial}{\partial z} + \frac{1}{a_2 - a_3}\frac{X(g(x, y, z))}{1 + g(x, y, z)}(y\frac{\partial}{\partial x} - z\frac{\partial}{\partial z}).$$

Proposition 1. *For $a \notin \mathcal{R}$, there exist a local diffeomorphism $\phi : z = (z_1, \cdots, z_{n+2}) \mapsto \tilde{z} = (\tilde{z}_1, \cdots, \tilde{z}_{n+2})$ such that the foliation $\mathcal{F}$ is given locally by $(\tilde{z}^{\gamma_0}, \cdots, \tilde{z}^{\gamma_{n+2}})$*

Proof. We just make a convenable change of variables. □

Open question. To complete the proof of Conjecture we must solve the nongeneric case $a = (a_1, \ldots, a_k) \in \mathcal{R}$ because in this case the rank of exponent matrix

$$m(a_1, \ldots, a_k) = \begin{pmatrix} \gamma_0^1 & \cdots & \gamma_0^{n+2} \\ \gamma_1^1 & \cdots & \gamma_1^{n+2} \\ \vdots & \vdots & \vdots \\ \gamma_n^1 & \cdots & \gamma_n^{n+2} \end{pmatrix}$$

is not maximal.

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